Worksheet: Elementary row operations, row-echelon form, and Gauss-Jordan
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Elementary row operations include the interchange of two equations (rows), multiplying a row by a non-zero constant, and adding a multiple of one row to another row.
Examples:

An exchange of rows R_1 and R_2:

$$\begin{bmatrix} 1 & 8 & -2 & 3 \\ 7 & 5 & 0 & 1 \\ 0 & -2 & 1 & 4 \end{bmatrix} \qquad R_1 \rightleftarrows R_2 \begin{bmatrix} 7 & 5 & 0 & 1 \\ 1 & 8 & -2 & 3 \\ 0 & -2 & 1 & 4 \end{bmatrix}$$

Multiplying row R_2 by the constant 2:

$$\begin{bmatrix} 1 & 8 & -2 & 3 \\ 7 & 5 & 0 & 1 \\ 0 & -2 & 1 & 4 \end{bmatrix} \qquad 2R_2 \rightarrow \begin{bmatrix} 1 & 8 & -2 & 3 \\ 14 & 10 & 0 & 2 \\ 0 & -2 & 1 & 4 \end{bmatrix}$$

Adding a multiple of R_3 to R_2:

$$\begin{bmatrix} 1 & 8 & -2 & 3 \\ 7 & 5 & 0 & 1 \\ 0 & -2 & 1 & 4 \end{bmatrix} \qquad \frac{1}{2}R_3 + R_2 \rightarrow \begin{bmatrix} 1 & 8 & -2 & 3 \\ 7 & 4 & \frac{1}{2} & 3 \\ 0 & -2 & 1 & 4 \end{bmatrix}$$

A matrix in *row-echelon form* consists of all zeros at the bottom of the matrix, underneath a "staircase" of leading 1s. A matrix is in *reduced row-echelon form* if every column with a leading 1 has a 0 in every other position.

Row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Here we are going to solve a system of equations by use elementary row operations to convert an augmented matrix into reduced row-echelon form. This method is called *Gauss-Jordan elimination.*

Rewrite the following systems of equations as an augmented matrix, and solve using Gauss-Jordan elimination:

$$\begin{cases} x-2y+3z=9 \\ -x+3y=-4 \\ 2x-5y+5z=17 \end{cases}$$

In augmented matrix form:

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

First, convert to row-echelon form:

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \qquad R_1+R_2 \rightarrow \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

$$-2R_1+R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \qquad R_2+R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$1/2\,R_3 \rightarrow \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

At this point, we can use *back-substitution* to solve. First, rewrite the augmented matrix as a system of equations:

$$\begin{cases} x-2y+3z=9 \\ y+3z=5 \\ z=2 \end{cases}$$

You can see that we've already found that $z=2$. Substituting 2 for z in the second equation, we get $y+3(2)=5$, or simply $y=-1$. Substituting both the values into the first equation, we get $x-2(-1)+3(2)=9$, or $x=1$. So our solution is $\{1,-1,2\}$.

But the question asked us to solve using Gauss-Jordan. So we must solve by converting to reduced row-echelon form:

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad 2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$-3R_3 + R_2 \rightarrow \begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad -9R_3 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Rewriting this as a system of equations, we get:

$$\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

That's it.